0.1 Example 1

In our first experiment, we consider a rather simple case on the unit cube $\Omega = (-1,1)^3$ in 3D with a homogeneous boundary condition that $\mathbf{u} = 0$ on $\partial\Omega$. The coefficient μ is kept constant (equal 1) all over the domain; the coefficient $\beta \in \{\pm 10^4, \pm 10^2, \pm 1, \pm 10^{-2}, \pm 10^{-4}\}$, and all the computations are conducted on uniformly refined (h-refinements) meshes. The smooth analytic solution $\mathbf{u}$ is given by

$$\mathbf{u}(x) = \mathbf{u}(x_1, x_2, x_3) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (1 - x_1^2)(1 - x_2^2)(1 - x_3^2). \tag{1}$$

The load function f can be calculated for the given solution $\mathbf{u}$.

Numerical results of exact and approximate errors, in H-curl and energy norms, together with the effectivity indices are shown in Tables 1 and 2 for positive values of β. And for negative values of β, we present the results in Tables 3 and 4.

In Tables 5 and 6 , we present the condition numbers of the diagonally-rescaled stiffness matrices for W and V with respect to the number of degrees of freedom on different mesh refinement levels for positive and negative values of β, respectively. For $\beta \in \{10^2, 10^4\}$, no results were obtained for condition numbers of both stiffness matrices for W and V. This is the error message:

Eigensystem: Preconditioner negative

Traceback (most recent call last):

File "<string>", line 204, in <module>

File "<string>", line 164, in CalcError

ValueError: max() arg is an empty sequence.

β	$Dof(V)$	$\|\mathbf{u}-\hat{\mathbf{u}}\|_{H(\mathrm{curl};\Omega)}$	$\|\varepsilon\|_{H(\mathrm{curl};\Omega)}$	order	EFF
10^{-4}	26	2.7446E+00	1.5085E+00		0.55
	180	3.4953E+01	1.7423E+00	-0.208	0.05
	1232	3.4110E+00	1.0144E+00	0.780	0.30
	9000	6.6006E-01	5.2400E-01	0.953	0.79
	68696	3.1056E-01	2.6283E-01	0.995	0.85
	536760	1.5508E-01	1.3108E-01	1.004	0.85
10^{-2}	26	2.7448E+00	1.5113E+00		0.55
	180	2.1244E+00	1.7423E+00	-0.205	0.82
	1232	1.1927E+00	1.0144E+00	0.780	0.85
	9000	6.1610E-01	5.2400E-01	0.953	0.85
	68696	3.1013E-01	2.6283E-01	0.995	0.85
	536760	1.5508E-01	1.3108E-01	1.004	0.85
1	26	2.7880E+00	2.2035E+00		0.79
	180	2.1044E+00	1.7765E+00	0.311	0.84
	1232	1.1946E+00	1.0221E+00	0.797	0.86
	9000	6.1652E-01	5.2519E-01	0.961	0.85
	68696	3.1019E-01	2.6299E-01	0.998	0.85
	536760	1.5509E-01	1.3111E-01	1.004	0.85
10^{2}	26	2.8476E+00	1.0156E+01		3.57
	180	5.1324E+00	8.2085E+01	-3.015	15.99
	1232	6.5658E+00	6.4160E+02	-2.966	97.72
	9000	6.1527E+01	1.2302E+04	-4.261	199.94
	68696	1.5129E+01	4.8518E+00	11.308	0.32
	536760	3.4120E+00	5.1703E-01	3.230	0.15
10^{4}	26	2.8325E+00	3.9530E+00		1.40
	180	2.8792E+00	4.6775E+00	-0.243	1.62
	1232	2.2254E+00	4.8109E+00	-0.041	2.16
	9000	1.8429E+00	1.1823E+01	-1.297	6.42
	68696	2.0705E+00	1.6842E+04	-10.476	8134.23
	536760	2.0789E+03	9.2369E+06	-9.090	4443.15

Table 1: Numerical errors, convergence order, and effectivity indices of the error estimator in H-curl norm when $\beta > 0$ (Uniform refinement).

β	$Dof(V)$	$\|\|\|\mathbf{u}-\hat{\mathbf{u}}\|\|\|$	$\|\|\|\varepsilon\|\|\|$	order	EFF
10^{-4}	26	2.2610E+00	1.4650E+00		0.65
	180	1.8890E+00	1.7101E+00	-0.223	0.91
	1232	1.0607E+00	1.0089E+00	0.761	0.95
	9000	5.4477E-01	5.2327E-01	0.947	0.96
	68696	2.7317E-01	2.6274E-01	0.994	0.96
	536760	1.3636E-01	1.3107E-01	1.003	0.96
10^{-2}	26	2.2663E+00	1.4677E+00		0.65
	180	1.8593E+00	1.7104E+00	-0.221	0.92
	1232	1.0616E+00	1.0089E+00	0.762	0.95
	9000	5.4552E-01	5.2329E-01	0.947	0.96
	68696	2.7356E-01	2.6274E-01	0.994	0.96
	536760	1.3655E-01	1.3107E-01	1.003	0.96
1	26	2.7880E+00	2.2035E+00		0.79
	180	2.1044E+00	1.7765E+00	0.311	0.84
	1232	1.1946E+00	1.0221E+00	0.797	0.86
	9000	6.1652E-01	5.2519E-01	0.961	0.85
	68696	3.1019E-01	2.6299E-01	0.998	0.85
	536760	1.5509E-01	1.3111E-01	1.004	0.85
10^{2}	26	1.4946E+01	1.9244E+01		1.29
	180	1.0855E+01	1.1605E+02	-2.592	10.69
	1232	1.0437E+01	9.0292E+02	-2.960	86.51
	9000	8.6621E+01	1.7313E+04	-4.261	199.87
	68696	2.1334E+01	5.2426E+00	11.689	0.25
	536760	4.8564E+00	5.2494E-01	3.320	0.11
10^{4}	26	1.4755E+02	1.1796E+02		0.80
	180	8.8793E+01	7.7682E+01	0.603	0.87
	1232	4.9855E+01	4.4785E+01	0.795	0.90
	9000	2.7027E+01	3.0039E+01	0.576	1.11
	68696	1.4340E+01	2.3963E+04	-9.640	1671.01
	536760	2.9399E+03	1.3066E+07	-9.082	4444.35

Table 2: Numerical errors, convergence order, and effectivity indices of the error estimator in energy norm when $\beta > 0$ (Uniform refinement).

β	$Dof(V)$	$\|\mathbf{u}-\hat{\mathbf{u}}\|_{H(\mathrm{curl};\Omega)}$	$\|\varepsilon\|_{H(\mathrm{curl};\Omega)}$	order	EFF
-10^{-4}	26	2.7446E+00	1.5084E+00		0.55
	180	3.4954E+01	1.7423E+00	-0.208	0.05
	1232	3.4111E+00	1.0144E+00	0.780	0.30
	9000	6.6006E-01	5.2400E-01	0.953	0.79
	68696	3.1056E-01	2.6283E-01	0.995	0.85
	536760	1.5508E-01	1.3108E-01	1.004	0.85
-10^{-2}	26	2.7443E+00	1.5056E+00		0.55
	180	2.1245E+00	1.7423E+00	-0.211	0.82
	1232	1.1927E+00	1.0143E+00	0.780	0.85
	9000	6.1610E-01	5.2399E-01	0.953	0.85
	68696	3.1012E-01	2.6283E-01	0.995	0.85
	536760	1.5508E-01	1.3108E-01	1.004	0.85
-1	26	2.7307E+00	1.4221E+00		0.52
	180	2.0934E+00	1.7601E+00	-0.308	0.84
	1232	1.1917E+00	1.0155E+00	0.794	0.85
	9000	6.1597E-01	5.2409E-01	0.954	0.85
	68696	3.1011E-01	2.6284E-01	0.996	0.85
	536760	1.5508E-01	1.3109E-01	1.004	0.85
-10^{2}	26	2.8195E+00	3.4857E+00		1.24
	180	2.5384E+00	3.3271E+00	0.067	1.31
	1232	1.5085E+00	2.3294E+00	0.514	1.54
	9000	7.5273E-01	1.3064E+00	0.834	1.74
	68696	3.4487E-01	5.6134E-01	1.219	1.63
	536760	3.4487E-01	5.6134E-01	1.219	1.63
-10^{4}	26	2.8323E+00	3.9382E+00		1.39
	180	2.8680E+00	4.5845E+00	-0.219	1.60
	1232	2.1727E+00	4.4108E+00	0.056	2.03
	9000	1.6218E+00	3.9907E+00	0.144	2.46
	68696	1.0480E+00	3.3564E+00	0.250	3.20
	536760	5.2710E-01	2.5741E+00	0.383	4.88

Table 3: Numerical errors, convergence order, and effectivity indices of the error estimator in H-curl norm when $\beta < 0$ (Uniform refinement).

β	$Dof(V)$	$\|\|\|\mathbf{u}-\hat{\mathbf{u}}\|\|\|$	$\|\|\|\varepsilon\|\|\|$	order	EFF
-10^{-4}	26	2.2610E+00	1.4649E+00		0.65
	180	1.8890E+00	1.7101E+00	-0.223	0.91
	1232	1.0607E+00	1.0089E+00	0.761	0.95
	9000	5.4477E-01	5.2327E-01	0.947	0.96
	68696	2.7317E-01	2.6274E-01	0.994	0.96
	36760	1.3636E-01	1.3107E-01	1.003	0.96
-10^{-2}	26	2.2662E+00	1.4631E+00		0.65
	180	1.8593E+00	1.7104E+00	-0.225	0.92
	1232	1.0616E+00	1.0089E+00	0.762	0.95
	9000	5.4552E-01	5.2328E-01	0.947	0.96
	68696	2.7356E-01	2.6274E-01	0.994	0.96
	536760	1.3655E-01	1.3107E-01	1.003	0.96
-1	26	2.7307E+00	1.4221E+00		0.52
	180	2.0934E+00	1.7601E+00	-0.308	0.84
	1232	1.1917E+00	1.0155E+00	0.794	0.85
	9000	6.1597E-01	5.2409E-01	0.954	0.85
	68696	3.1011E-01	2.6284E-01	0.996	0.85
	536760	1.5508E-01	1.3109E-01	1.004	0.85
-10^{2}	26	1.4957E+01	1.1450E+01		0.77
	180	9.2302E+00	7.5840E+00	0.594	0.82
	1232	5.2473E+00	3.9166E+00	0.953	0.75
	9000	2.8454E+00	1.7176E+00	1.189	0.60
	68696	1.4782E+00	6.1985E-01	1.470	0.42
	536760	7.4902E-01	2.0610E-01	1.589	0.28
-10^{4}	26	1.4755E+02	1.1775E+02		0.80
	180	8.8794E+01	7.7320E+01	0.607	0.87
	1232	4.9853E+01	4.4005E+01	0.813	0.88
	9000	2.7012E+01	2.4074E+01	0.870	0.89
	68696	1.4178E+01	1.2482E+01	0.948	0.88
	536760	7.2709E+00	5.9796E+00	1.062	0.82

Table 4: Numerical errors, convergence order, and effectivity indices of the error estimator in energy norm when $\beta < 0$ (Uniform refinement).

β	$Dof(V)$	Cnd. No (V)	Cnd. No (W)
10^{-4}	26	79999.289	36.076
	180	422578.963	63.754
	1232	2213695.104	80.113
	9000	915037.942	90.301
	68696	169107.658	93.435
	536760	194118.539	95.012
10^{-2}	26	799.289	36.252
	180	4132.533	63.827
	1232	22021.853	80.125
	9000	111247.673	90.304
	68696	281605.782	93.436
	536760	218562.354	95.013
1	26	7.279	89.649
	180	36.277	72.875
	1232	166.470	81.341
	9000	837.108	90.644
	68696	4142.714	93.522
	536760	18690.529	95.034
10^2			
10^4			

Table 5: Condition numbers of the stiffness matrices for V and W when $\beta > 0$ (Uniform refinement).

β	$Dof(V)$	Cnd. No (V)	Cnd. No (W)
-10^{-4}	26	80000.711	36.072
	180	419178.450	63.779
	1232	2160434.527	84.151
	9000	720265.625	91.202
	68696	170300.780	94.651
	536760	193680.976	95.012
-10^{-2}	26	800.711	35.898
	180	5553.342	63.707
	1232	21701.830	84.140
	9000	88097.925	91.199
	68696	121484.607	94.650
	536760	174241.066	95.012
-1	26	8.706	25.655
	180	64.518	57.751
	1232	343.103	83.063
	9000	1188.105	90.886
	68696	4221.520	94.566
	536760	15901.549	94.991
-10^{2}	26	3.027	52.782
	180	3.759	50.786
	1232	9.478	50.106
	9000	35.606	71.847
	68696	133.511	87.691
	536760	364.377	92.964
-10^{4}	26	3.513	75.309
	180	4.727	129.008
	1232	6.068	136.044
	9000	7.152	108.870
	68696	7.687	72.568
	536760	11.437	51.688

Table 6: Condition numbers of the stiffness matrices for V and W when $\beta < 0$ (Uniform refinement).